"CLASSIFYING" LIE ALGEBRAS

Lecture 6

including Roots and weights etc.

[Knapp: Lie Groups Beyond an Introduction, I.5–I.8, II.1–II.5]
[Laksh-Brown Chap 7].

Lie algebras: Recall ideal ≈ normal subgroup (think "normal subalg")

- **abelian**: $[x,y]=0 \ \forall x,y$.
- **solvable**: inductively built from abelian ones...
- **semisimple**: no solvable ideals (except $\{0\}$)
- **simple**: no ideals at all (except $\{0\}$).

Ex. $\mathfrak{b} = \left\{ \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \right\}$ solvable as $[\mathfrak{b},\mathfrak{b}] = D'\mathfrak{b} = \begin{pmatrix} 0 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{pmatrix}$ $[D'\mathfrak{b}, D'\mathfrak{b}] = \begin{pmatrix} 0 & 0 & * \\ & \ddots & 0 \\ & & 0 \end{pmatrix}$

Ex. $\mathfrak{gl}_n\mathbb{C} = \{(*)\}$ is not semisimple. $\{cI_{n\times n}, c\in k\}$ is a nonzero abelian ideal (in particular solvable).
$\mathfrak{gl}_n\mathbb{R}$ //

Exercise: $\mathfrak{sl}_n\mathbb{C}$, $\mathfrak{sl}_n\mathbb{R}$ are simple. $(\mathfrak{sl}_n k = \{(*) \in \mathfrak{gl}_n k \mid \text{trace}=0\})$

Theorem: semisimple $\Leftrightarrow$ direct sum of simple.
Pf. See Knapp.

Ideals $\mathfrak{a} \subset \mathfrak{g}$ mean $\mathfrak{g}/\mathfrak{a}$ inherits a Lie bracket (like G/N group!)

So the overall idea is to understand Lie alg as:

$$\underbrace{\mathfrak{g}}_{\text{unknown}} \rightsquigarrow \quad 0 \to \underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{solvable}} \longrightarrow \mathfrak{g} \longrightarrow \underbrace{\mathfrak{g}/\mathrm{rad}(\mathfrak{g})}_{\text{semisimple}} \to 0$$

} extension (≈ trivial Jordan Decomp)

solvable: iterated extension of abelian.
semisimple: direct sum of simple

So an understanding of simple ones, abelian ones, and extensions would suffice.

Simple Lie alg over $\mathbb{C}$ admit a beautiful classification
(W. Killing 1890, E. Cartan 1894)

Idea: Take $\mathfrak{g}$ apart and recognize canonical "pieces". The classification is in terms of the combinatorics of how they fit together.

It's a little complicated so let's do all the steps for $\mathfrak{sl}_n\mathbb{C}$ first.

Let $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C} = \{$traceless $n\times n$ complex matrices w/ commutator$\}$

let $\mathfrak{h} = \{$diagonal matrices in $\mathfrak{g}\}$

$\mathfrak{a} = \{$ real diag matrices in $\mathfrak{g}\}$

as Lie alg!

so: $\underbrace{\overbrace{\mathfrak{a}}^{\text{real}} \subset \overbrace{\mathfrak{h} \subset \mathfrak{g}}^{\text{complex}}}_{\text{abelian.}}$ subalgebras, also $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a}$.

Thus $\mathfrak{h}$ is the Lie algebra complexification of $\mathfrak{a}$, $\mathfrak{h} = (\mathfrak{a})^{\mathbb{C}}$ or equivalently we say $\mathfrak{a}$ is a real form of $\mathfrak{h}$.

Given $x \in \mathfrak{g}$, we get a linear map $ad_x : \mathfrak{g} \to \mathfrak{g}$ by $y \mapsto [x,y]$. } adjoint action of x

Now if $t \in \mathfrak{h}$, ad_t is diagonalizable, and a basis that diagonalizes it is:

E_{ij} = matrix w/ 1 in (i,j) else zero. ($\in \mathfrak{h}$ iff $i=j$)

Then we can make a basis of $\mathfrak{g}$ from one of $\mathfrak{h}$ plus

$\{E_{ij} \mid i \neq j\}$.

$ad_t(E_{ij}) = (t_i - t_j)E_{ij}$ as one can check, if $t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$

So no matter what t is, $\{E_{ij}\}$ + basis of $\mathfrak{h}$ gives a basis of eigenvectors. But to say what the eigenvalue is, you need to know t.

But the map (element $t \in \mathfrak{h}$) $\longmapsto$ (the eigenval of ad_t on E_{ij}) is linear, so it defines an element of $\mathfrak{h}^* = Hom(\mathfrak{h}, \mathbb{C})$

Def. An eigenvector of $ad_{\mathfrak{h}}$ w/ eigenvalue $\lambda \in \mathfrak{h}^*$ means a vector $x \in \mathfrak{g}$ s.t.

$$\underset{\substack{\cap \\ \mathfrak{h}.}}{ad_t}(x) = \underbrace{\lambda(t)}_{\text{in } \mathbb{C}} x$$

But those names are not used. They would make too much sense. Instead:

Def. $x \in \mathfrak{g} - \{0\}$ is called a weight vector with weight $\lambda \in \mathfrak{h}^* - \{0\}$ if $ad_t(x) = \lambda(t) x \quad \forall\, t \in \mathfrak{h}$.

Let $e_i \in \mathfrak{h}^*$ be the map $\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \longmapsto t_i$

Then E_{ij} is a weight vector with weight $e_i - e_j$.

Def. The weights that occur for weight vectors in $\mathfrak{g}$ (the "spectrum" of $\mathfrak{h}$ on $\mathfrak{g}$) are called <u>roots</u>.

The roots of $\mathfrak{g} = \mathfrak{sl}_n\mathbb{C}$ are $\{e_i - e_j \mid i \neq j\} = \Phi \subset \mathfrak{h}^*$

Thus $\boxed{\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j}}$ where $\mathfrak{g}_\alpha$ is set of all weight vectors for $\alpha \in \mathfrak{h}^*$.

root space decomposition

$$\begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \qquad \mathbb{C} \cdot E_{ij} \quad \begin{pmatrix} \cdots & * & \\ & \vdots & \end{pmatrix}$$

Note Φ spans $\mathfrak{h}^*$ over $\mathbb{C}$.

Nice property: $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \begin{cases} \mathfrak{g}_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\ \{0\} & \text{if } \alpha + \beta \notin \Phi, \text{ is not } 0 \\ \mathfrak{h} & \text{if } \alpha + \beta = 0. \end{cases}$

Ex. $n=3$: $\left[\underset{e_1-e_2}{\begin{pmatrix}0&1&0\\&&0\\&&0\end{pmatrix}}, \underset{e_1-e_3}{\begin{pmatrix}0&0&1\\&&0\\&&0\end{pmatrix}}\right] = 0$ as $2e_1-e_2-e_3$ not root

$$\left[\underset{e_1-e_2}{\begin{pmatrix}0&1&0\\&&\\&&\end{pmatrix}}, \underset{e_2-e_1}{\begin{pmatrix}0&&\\1&&\\0&&\end{pmatrix}}\right] = \underset{\text{in } \mathfrak{h}}{\begin{pmatrix}1&0&\\0&-1&\\&&0\end{pmatrix}}$$

$$\left[\underset{e_1-e_2}{\begin{pmatrix}0&1&0\\&&0\\&&0\end{pmatrix}}, \underset{e_2-e_3}{\begin{pmatrix}&&0\\0&0&1\\&&0\end{pmatrix}}\right] = \underset{e_1-e_3.}{\begin{pmatrix}0&0&1\\&&0\\&&0\end{pmatrix}} \qquad \alpha+\beta\in\Phi.$$

On $\mathfrak{a}$, each root is $\mathbb{R}$-valued, and restr to $\mathfrak{a}$ determines it.

So $\alpha\in\Phi$ can be considered as elt of $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h},\mathbb{C})$
OR elt of $\mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a},\mathbb{R})$

Now we want to introduce a notion of positivity on $\mathfrak{a}^*$, s.t.

① If $\varphi\in\mathfrak{a}^*$ is nonzero, one of $\varphi,-\varphi$ is positive (exactly one!)

② The set of positive elts is closed under $+$ and $\mathbb{R}^+$-mult.

$\mathfrak{a}^* = \mathrm{Hom}(\mathfrak{a},\mathbb{R}) = \mathrm{Hom}(\{(t_1\ldots t_n) \mid \Sigma t_i = 0\}, \mathbb{R})$ $\qquad \sigma(t_1\ldots t_n) = \Sigma t_i$

$= \mathrm{Hom}(\ker\sigma, \mathbb{R})$

$= (\mathbb{R}^n)^* / \mathrm{span}\,\sigma$

i.e. $\mathfrak{a}^* = \mathrm{span}(e_1\ldots e_n)$ but they're not quite indep as $e_1+\cdots+e_n = \sigma$ which is zero on $\mathfrak{a}$.

But any element of $\mathfrak{a}^*$ can be uniquely rep as $\Sigma a_i e_i$ where $\Sigma a_i = 0$.

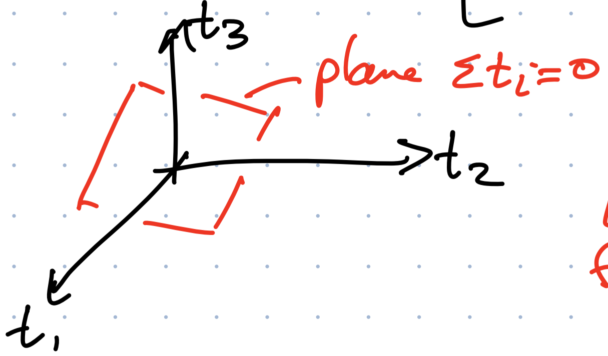
e.g. $e_1 = \frac{2}{3}e_1 - \frac{1}{3}e_2 - \frac{1}{3}e_3$ on $\mathfrak{a}\subset\mathfrak{sl}_3\mathbb{C}$.

So let's say $\varphi \in \mathfrak{a}^*$ is positive in this case if $\varphi = \sum a_i e_i$ with $\sum a_i = 0$ and the first nonzero coef a_k is positive

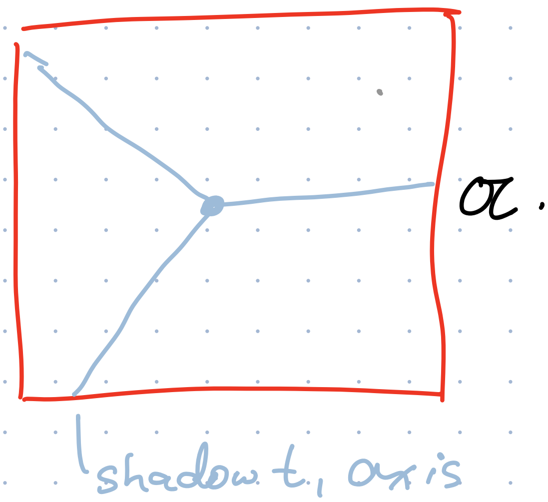
(i.e. φ is "lexicographically greater than zero").

Let's try to make a picture.

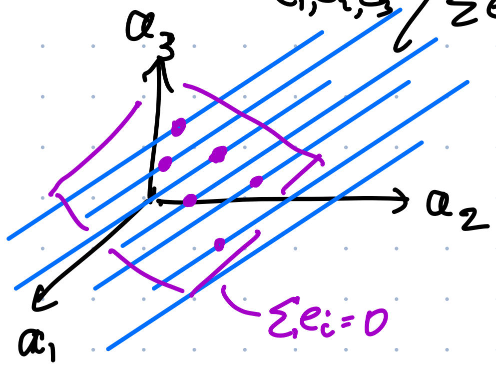
$$\mathfrak{g} = \mathfrak{sl}_3\mathbb{C} \qquad \mathfrak{a} = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \,\middle|\, \sum t_i = 0 \right\} \subset \mathbb{R}^3.$$



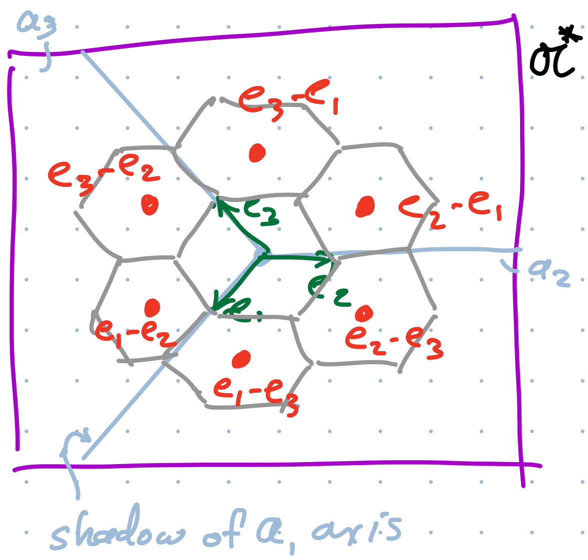
$\rightsquigarrow$ Look at it from huge $\cdot (1,1,1)$

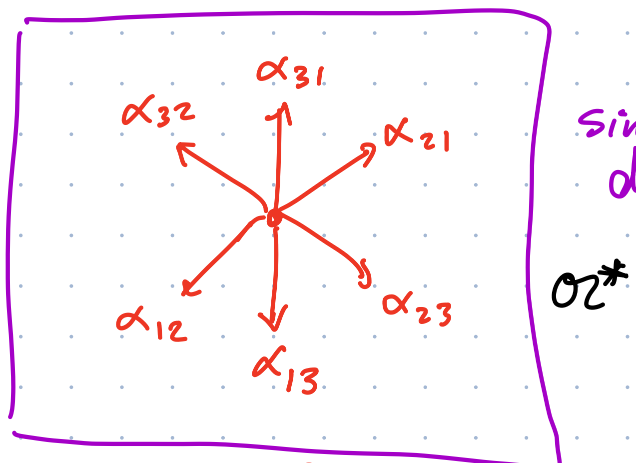


Now $\mathfrak{a}^* = \mathbb{R}^3_{e_1,e_2,e_3} / \sum e_i$ $\qquad$ $\sum a_i e_i$ generic



$\rightsquigarrow$ intersect any translate of span$(e_1+e_2+e_3)$ w/ $\sum a_i = 0$.





simplified drawing

Roots of $\mathfrak{sl}_3\mathbb{C}$

__Exercise__: shade the positive region!